

Tests of Conformal Field Theory at the Yang-Lee Singularity

TOMASZ WYDRO

*Statistical Physics Group, P2M Dpt, Institut Jean Lamour
Nancy Université, Université Paul Verlaine - Metz
BP 70239, 54506 Vandoeuvre-les-Nancy Cedex, France
wydro@lpm.u-nancy.fr*

JOHN F. McCABE

*2331 Gales Court, Scotch Plains, NJ 07076, USA
jfmccabe@lycos.com*

December 18, 2009

Abstract

This paper studies the Yang-Lee edge singularity of 2-dimensional ($2D$) Ising model based on a quantum spin chain and transfer matrix measurements on the cylinder. Based on finite-size scaling, the low-lying excitation spectrum is found at the Yang-Lee edge singularity. Based on transfer matrix techniques, the single structure constant is evaluated at the Yang-Lee edge singularity. The results of both types of measurements are found to be fully consistent with the predictions for the (A_4, A_1) minimal conformal field theory, which was previously identified with this critical point.

1 1. Introduction

In 1978, Fisher [1] proposed that Yang-Lee edge singularities [2, 3] are critical points. Later, Cardy [4] argued that the Yang-Lee edge singularity of the 2D Ising model should be identified with the (A_4, A_1) minimal conformal field theory (CFT) [5, 6] of the ADE classification [7]. Cardy's identification provides CFT predictions for this Yang-Lee edge singularity.

This article tests different predictions coming from Cardy's identification.

In section 2, we provide measurements of the low-lying excitation spectrum at Yang-Lee edge singularity of the 2D Ising model. The measured low-lying excitation spectrum is also compared with predictions from Cardy's identification of the (A_4, A_1) minimal CFT with this Yang-Lee edge singularity of the 2D Ising model [4, 8, 9].

Cardy's identification also determines the forms of 2-point and 3-point correlations. In particular, these correlations define universal amplitudes, which are known as structure constants [5, 10]. Such predictions are an important advance that CFT brought to the understanding of critical points of 2D statistical models. No tests of such predictions have been performed for critical points associated with non-unitary CFTs.

In section 3, we provide a measurement of the universal amplitude associated with the Yang-Lee edge singularity of the 2D Ising model. The measured amplitude is also compared with the prediction from Cardy's identification of the (A_4, A_1) minimal CFT with this Yang-Lee edge singularity.

2. Excitation Spectrum at the Yang-Lee Edge Singularity of the 2D Ising model

The 2D Ising model in an imaginary external magnetic field is associated with a quantum spin chain whose Hamiltonian, H_{Ising} , on an N -site chain, is given by [15]:

$$H_{Ising} = - \sum_{n=1}^N \{t\sigma_z(n)\sigma_z(n+1) + iB\sigma_z(n) + \sigma_x(n)\}. \quad (1)$$

In Eq. (1), $\sigma_x(n)$ and $\sigma_z(n)$ are Pauli spin matrices at the site n , parameter " t " is a positive coupling for a ferromagnetic spin-spin interaction, and iB is a purely imaginary external magnetic field. In Eq. (1), the last term produces inter-row single spin flips in the associated 2D transfer matrix [16, 17].

Below, the phenomenological renormalization group (PRG) is used to determine critical values of imaginary magnetic field, $iB_{YL}(N)$, for various lengths, N , of the chain. For imaginary magnetic fields, the PRG equation requires that:[11, 8]

$$[N-1]m(B_{YL}(N), N-1) = [N]m(B_{YL}(N), N). \quad (2)$$

In Eq. (2), $m(B, N) = [E_1(B, N) - E_0(B, N)]$ where $E_0(B, N)$ and $E_1(B, N)$ are energies for the ground state "0" and the first excited state "1" on a chain of length N . Below, $m(B, N)$ is referred to as $Gap(B, N)$ or more simply as $Gap(N)$. At these $B_{YL}(N)$'s, the Ising quantum spin chain exhibits the finite-size scaling behavior of the Yang-Lee edge singularity. In particular, if the $B_{YL}(N)$'s converge to a nonzero value as $N \rightarrow \infty$, that value will be the critical point for the Yang-Lee edge singularity of the 2D Ising spin model.

At these $B_{YL}(N)$'s, excitation energies should scale. In particular, CFT predicts how these energies will scale with the length, N , of the chain. For an excited energy eigenstate "i" of the quantum spin chain, an excitation energy, $E_i(N) - E_0(N)$, will scale as:[14]

$$E_i(N) - E_0(N) = \zeta 2\pi \frac{\Delta_i + \bar{\Delta}_i - (\Delta + \bar{\Delta})}{N}. \quad (3)$$

In Eq. (3), Δ_i and $\bar{\Delta}_i$ are left and right conformal dimensions of conformal field "i", and Δ and $\bar{\Delta}$ are conformal dimensions of the primary field having the lowest "negative" scaling dimension in the relevant non-unitary CFT. In Eq. (3), the constant ζ is non-universal, e.g., depending on the normalization of the Hamiltonian¹.

In minimal CFTs, the modular invariant forms of the partition functions [7, 12] determine the low-lying excitation spectrum and the central charges. For the (A_4, A_1) minimal CFT, Table 1 gives the energies of the low-lying excitations and degeneracies thereof as obtained from the associated partition function. Table 1 provides normalized excitation energies, which are ratios. For a state "i",

CFT	(A_4, A_1)					
Normalized Energies	0	1	2.5	5.0	6.0	7.5
Degeneracy	1	1	2	3	2	4

Table 1: Lowest excitations of (A_4, A_1) CFT

the normalized excitation energy is the ratio is the excitation energy of the state "i" over the excitation energy of the lowest excited state "1". Here, excitation energies are with respect to the ground state. The normalized excitation energies of Table 1 do not depend on non-universal constants such as ζ .

The critical magnetic fields, $B_{YL}(N)$, were obtained by solving the PRG eq. (2) for chains of different lengths. For these solutions, state energies were obtained by using the Lanczos algorithm for H_{Ising} of eq. (1). Table 2 shows critical fields, i.e., $B_{YL}(N)$'s, ground state energies, and lowest excitation energies, i.e., $Gap(N)$'s. These measurements were obtained for Ising quantum spin chains in which the coupling, t , is 0.1. Table 2 shows that $NxGap(N)$ scales to a constant as $N \rightarrow \infty$ as expected from the PRG.

The critical magnetic field values, i.e., the B_{YL} 's, were used to find the low-lying excitation spectra of Ising quantum spin chains of various lengths. Table 3 provides measured spectra including both energies and degeneracies. Here, excitation energies are also normalized by dividing by the lowest excitation energy, i.e., as already described to remove any dependence on the non-universal constant ζ .

Figures 1 - 4 plot the measured excitation energies of the states A - F as a function of the inverse of the length of the Ising quantum spin chain.

A visual inspection of Figures 1 - 4 shows that the type A, B, C, D, E, and F states form four distinct sets A, B & C, D, and E & F. Within each set, the states have energies that approach the same value as $1/N \rightarrow 0$. The excitation energies of the states of sets A, B & C, D, and E & F approach about 2.45, 5.0, 6.03, and 7.6, respectively, as $1/N \rightarrow 0$. A BST analysis shows that excitation energies of the type A, B, C, D, E, and F states scale to 2.4995(5), 5.005(1), 5.003(3), 5.99(1), 7.54(8), and 7.60(7), respectively, in this limit. These PRG measurements of the low-lying excitation energies and degeneracies agree well with the predictions for the (A_4, A_1) CFT as in Table 1.

¹ ζ is the "sound velocity" in the dispersion relation of the critical Hamiltonian

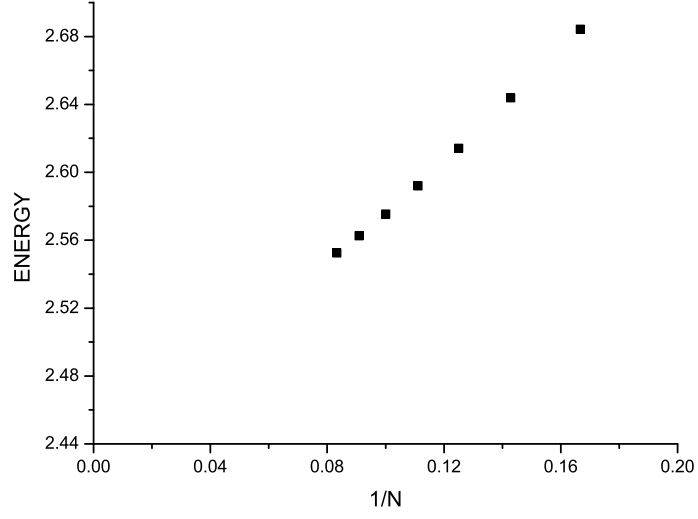


Figure 1: Energies of the type A states as a function of $1/N$.

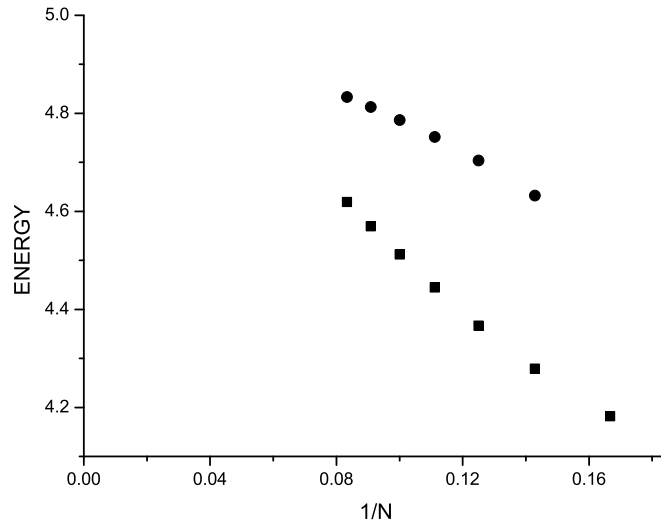


Figure 2: Energies of type B states (squares) and type C states (circles) as a function of $1/N$.

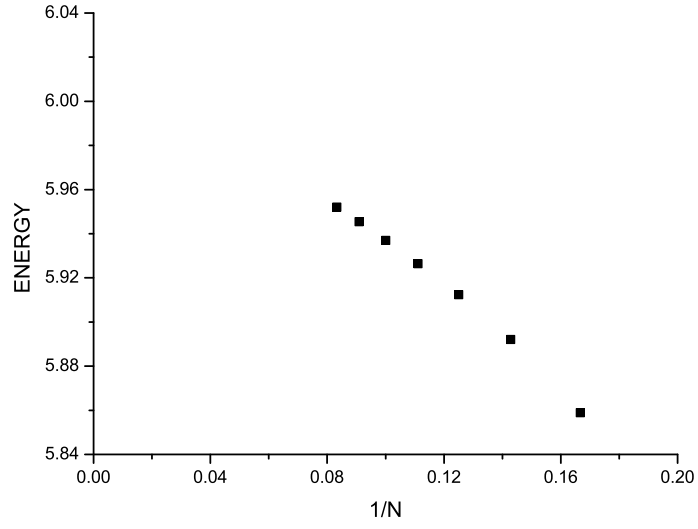


Figure 3: Energies of type D states as a function of $1/N$.

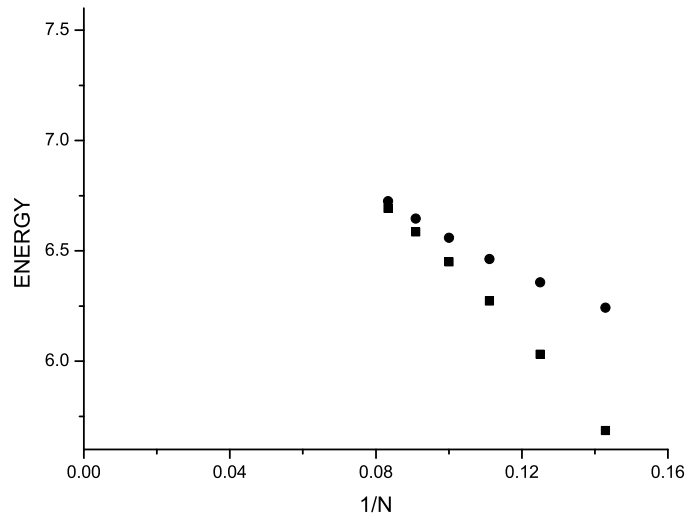


Figure 4: Energies of type E states (squares) and F states (circles) as a function of $1/N$.

Number of Sites	$B_{YL}(N)$	Energy of ground state	Gap(N)	$N \times \text{Gap}(N)$
3	.2459180i	-2.8811043	.8103423	2.4310
4	.2384127i	-3.8028211	.6629112	2.6516
5	.2352339i	-4.7341982	.5613016	2.8065
6	.2337637i	-5.6688215	.4858628	2.9152
7	.2330279i	-6.6048003	.4275400	2.9928
8	.2326347i	-7.5414746	.3811698	3.0494
9	.2324118i	-8.4785910	.3435105	3.0916
10	.2322793i	-9.4160213	.3123765	3.1237
11	.2321972i	-10.353696	.286250	3.1488
12	.2321442i	-11.291568	.264041	3.1685
...
∞	.23193i	$-\infty$	0.0	3.2840

Table 2: Measurements of $B_{YL}(N)$, Ground state energy, Gap , and $N \times \text{Gap}$ for various chain lengths, N

State / [Degeneracy]	6	7	8	9	10	11	12
A / [2]	2.68432	2.64386	2.61415	2.59207	2.57540	2.56260	2.55253
B / [1]	4.18193	4.27896	4.36713	4.44474	4.51197	4.56977	4.61912
C / [2]	4.51738	4.63236	4.70368	4.75182	4.78652	4.81281	4.83329
D / [2]	5.85889	5.89208	5.91240	5.92644	5.93703	5.94544	5.95210
E / [2]	—	5.68559	6.03104	6.27270	6.45018	6.58573	6.69223
F / [2]	—	6.24252	6.35798	6.46344	6.55966	6.64694	6.72535

Table 3: Normalized excitation energies and degeneracies of lowest excited states A - F for Ising quantum spin chains with 6 to 12 sites.

3. Structure Constant at the Yang-Lee Edge Singularity

The non-unitary (A_4, A_1) minimal CFT has one primary field $\phi(z, \bar{z})$ with left and right conformal weights $-1/5$ and scaling dimension x of $-2/5$ [4]. For this field, $\phi(z, \bar{z})$, 2-point and 3-point correlations have the forms:

$$G_{\phi\phi}(z_1, \bar{z}_1, z_2, \bar{z}_2) = |z_1 - z_2|^{4/5}, \quad (4)$$

and

$$G_{\phi\phi\phi}(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3) = C|(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^{2/5}. \quad (5)$$

Cardy showed that the structure constant, C , of the non-unitary (A_4, A_1) minimal CFT is given by:[4]

$$C = \sqrt{-\frac{[\Gamma(6/5)]^2 \Gamma(1/5) \Gamma(2/5)}{\Gamma(3/5) [\Gamma(4/5)]^3}} \quad (6)$$

Below, numerical measurements at the Yang-Lee edge singularity are presented for this CFT prediction. The numerical measurements were made for the 2D Ising model, i.e., rather than for a spin chain. The 2D Ising model has a Hamiltonian H , given by:

$$H = - \sum_{j=1}^M \sum_{i=1}^N [J(S_{i,j}S_{i,j+1} + S_{i,j}S_{i+1,j}) + hS_{i,j}]. \quad (7)$$

The spin-spin coupling J is positive. In this model, the Yang-Lee edge singularity occurs above the critical temperature for a purely imaginary values of the magnetic field, h , i.e., $h = iB$ with B real [2, 3]. Below, the spin correlations were measured at a temperature, T , for which $J/k_B T = 0.1$.

The transfer matrix was used to measure correlation 2-spin and 3-spin correlations on torii of length, M , and diameters, N . In these evaluations, M was much larger than N , i.e., $M = 512$ and $N = 3 - 8$, so that correlations had distance behaviors for infinitely long cylinders at field separations small compared to M .

Finite-size scaling enabled the extraction of physical properties in the thermodynamic limit [13]. In particular, the spin correlations were measured at purely imaginary magnetic field values, $h(N) = iB_{YL}(N)$. Each value, $B_{YL}(N)$, satisfied the phenomenological renormalization group (PRG) equation for infinite cylinders of diameters $(N - 1)$ and N :

$$\frac{\xi(iB_{YL}(N), N - 1)}{N - 1} = \frac{\xi(iB_{YL}(N), N)}{N}. \quad (8)$$

In the PRG equation, $\xi(iB, N)$ is the spin-spin correlation length on the infinite cylinder of diameter N at the magnetic field iB ².

On a cylinder of width N , CFT predicts that correlations depend exponentially on distances between fields when said distances are large compared the cylinder's diameter, N [14]. When $|y_1 - y_2| \gg N$, the 2-point correlation of fields of scaling dimension, x , has the form $\exp(-2\pi x(y_1 - y_2)/N)$ where y_1 and y_2 are the positions of the fields along the axis of the infinite cylinder. For the 3-point correlation, the exponential behavior on the distances between the fields of the correlation is also determined by the scaling dimensions of the fields therein.

At the Yang-Lee edge singularity, amplitudes of 2-spin and 3-spin correlations, i.e., A_{ss} and A_{sss} , were used to evaluate the 3-spin structure constant. The values of the 3-spin structure constant, $C(N)$, were obtained from the relation:

$$C(N) = \frac{A_{sss}(iB_{YL}(N))}{[A_{ss}(iB_{YL}(N))]^{3/2}}. \quad (9)$$

In the above equation, $A_{ss}(iB_{YL}(N))$ and $A_{sss}(iB_{YL}(N))$, are amplitudes of the respective 2-spin and 3-spin correlations at PRG values for the magnetic field.

² ξ is measured by the first inverse gap (see [16])

PRG measurements of the correlation length $\xi(N)$ also provide a measurement of the conformal dimension, x , of the spin field, i.e., $x(N) = N/[2\pi\xi(iB_{YL}(N))]$. Scaling behaviors of these quantities with the cylinder's width, N , were used to obtain the values of the quantities as $N \rightarrow \infty$.

Table 4 provides our transfer matrix measurements³ for $M = 512$ and $J/k_B T = 0.1$.

N	$B_{YL}(N)$	$x(N)$	$ C(N) $
3	0.184802	0.353929	1.80838
4	0.183348	0.376870	1.83711
5	0.183064	0.385748	1.85736
6	0.182982	0.390108	1.87054
7	0.182951	0.392693	1.87937
8	0.182946	0.392911	1.88633
...
∞	...	0.398(2)	1.923(13)
<i>CFT</i>	—	0.4	1.9113

Table 4: PRG Measured values of conformal dimension and structure constant.

In Table 4, the ∞ line extrapolates the measured values to the thermodynamic limit, i.e., $N = \infty$. The extrapolated values were obtained from fits of the measured $x(N)$'s and $|C(N)|$'s to functions of form $f(N) = f(\infty) + f_1 N^{-\alpha}$, i.e., leading finite-size scaling forms.

In Table 4, the last line gives the predictions for x and $|C|$ from the (A_4, A_1) non-unitary minimal CFT model.

Figure 5 shows measurements of the structure constant $|C(N)|$ and a best fit (line), which accounts for a correction in $N^{-1.2}$. In Figure 5, the black squares are the measured $C(N)$'s, the empty square is the value of $C(\infty)$ from the best fit, and the black circle is the CFT prediction. Our finite-size scaling measurements produce a value for the structure constant that again agrees well with the prediction of the (A_4, A_1) minimal CFT.

³For $N = 3 - 7$, $B_{YL}(N)$'s were evaluated from the PRG equation. For $N = 8$, $B_{YL}(8)$ ' was estimated from the $B_{YL}(N)$'s for $N = 3 - 7$ by assuming a leading finite-size scaling behavior.

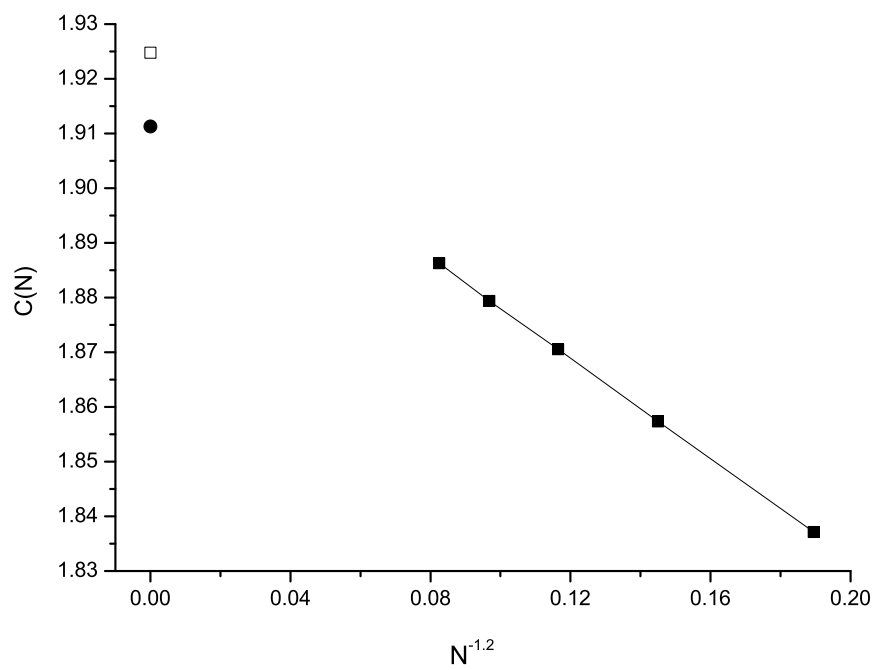


Figure 5: The measured structure constant (squares) for cylinders of diameter N plotted against nonlinear fit for $N = 4 - 8$.

References

- [1] M.E. Fisher, Phys. Rev. Lett. 40 (1978) 1610; see also D.A. Kurtz and M.E. Fisher, Journ. of Stat. Phys. 19 (1978) 205; Phys. Rev. B 20 (1979) 2785.
- [2] C.N. Yang, T.D. Lee, Phys. Rev. 87 (1952) 404.
- [3] C.N. Yang, T.D. Lee, Phys. Rev. 87 (1952) 410.
- [4] J.L. Cardy, Phys. Rev. Lett. **54**, 1354 (1985).
- [5] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333; for a review see, e.g.: M. Henkel, Conformal Invariance and Critical Phenomena (Springer-Verlag, Germany 1999); P. Di Francesco, P. Mathieu, and D. Sénéchal, Conformal Field Theory (Springer-Verlag, USA 1997) pages 210 - 228 .
- [6] D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984), and Comm. Math. Phys. **107**, 535 (1986).
- [7] A. Cappelli, C. Itzykson, J.-B. Zuber, Nucl. Phys. B 280 (1987) 445; Comm. Math. Phys. 113 (1987) 1; A. Kato, Mod. Phys. Lett. A 2 (1987) 585; for a review see e.g., C. Itzykson and J.-M. Drouffe, Statistical Field Theory (Cambridge University Press, U.K. 1989) chapter IX (1989).
- [8] C. Itzykson, H. Saleur, and J.-B. Zuber, Europhys. Lett. 2 (1986) 91.
- [9] K. Uzelac and R. Jullien, J. Phys. A 14 (1981) L151.
- [10] V.I.S. Dotsenko and V.A. Fateev, Nucl. Phys. **B240**, 312 (1984), and **B251**, 691 (1985).
- [11] M.P. Nightingale, Physica 83 A (1976) 561; B. Derrida and L. de Seze, J. Physique 43 (1982) 475; see, e.g., M.P. Nightingale, in Finite Size Scaling and Numerical Simulation of Statistical Systems (World Scientific Publishing, Singapore, Ed. V. Privman 1990) Ch. VII.
- [12] A. Rocha-Caridi in Vertex Operators in Mathematics and Physics MSRI Publications No. 3 (Springer, USA, Eds. J. Lepowski, S. Mandelstam and I.M. Singer 1985) 451.
- [13] M.E. Fisher and M.N. Barber, Phys. Rev. Lett. 28 (1972) 1516; for a brief review see, e.g., M. Henkel, Conformal Invariance and Critical Phenomena (Springer-Verlag, Germany 1999) chapter 3.
- [14] J.L. Cardy, J. Phys. **A17**, L385 (1984), and Nucl. Phys. **B270**, 186 (1986); for a review see M. Henkel, *Conformal Invariance and Critical Phenomena* (Springer-Verlag, Germany, 1999) Ch. 13.

- [15] M. Suzuki, Prog. Theor. Phys. 56 (1976) 1454; G. von Gehlen, V. Rittenberg, and T. Vescan, J. Phys. A 20 (1987) 2577; for a review see e.g., M. Henkel in Conformal Invariance and Critical Phenomena (Springer-Verlag, Germany 1999) Chs. 8-10.
- [16] J.B. Kogut, Rev. Mod. Phys. 51 (1979) 659.
- [17] See, e.g., the review of T. Wydro and J. McCabe, in Proceedings of the 7th International School on Theoretical Physics "Symmetry and Structural Properties of Condensed Matter" (World Scientific Publishing, Singapore, Eds. T. Lulek, B. Lulek, and A. Wal Singapore 2003) 9.
- [18] T. Wydro and J. McCabe, Int. J. of Mod. Phys. B 19 (2005) 3021.